sin $IHL=\frac{1}{3}\sqrt{3}$, cos $IHL=\frac{1}{3}\sqrt{6}$.
sin $EHL=\frac{1}{3}$, cos $EHL=\frac{2}{3}\sqrt{2}$.
$HI:HL=\sin ILH:\sin HIL=\sin EHL:\sin IHE$. $\therefore HI:w=\frac{1}{3}:\frac{1}{3}\sqrt{6}$ or $HI=w/\sqrt{6}=w\sqrt{6}/6$; $HE:w=\sin IHL:\sin HIL$; $HE:w=\frac{1}{3}\sqrt{3}:\frac{1}{3}\sqrt{6}$, or $HE=w/\sqrt{2}=w\sqrt{2}/2$.

Also solved by G. W. Greenwood.

AVERAGE AND PROBABILITY.

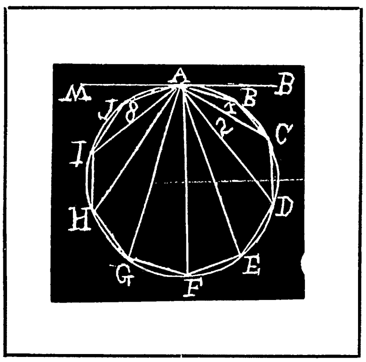
163. Proposed by R. D. CARMICHAEL, Anniston, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle?

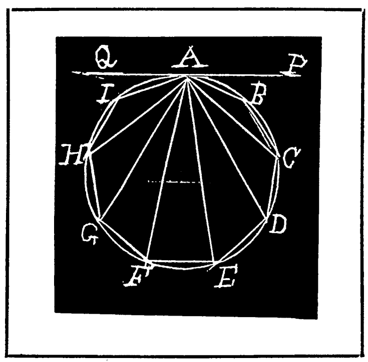
Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Whether n be even or odd, there can be formed, at any vertex, as B, or C, or D, etc., by joining it with A and any other vertex, $(n-2)$ triangles. Since there are n vertices, the total number of triangles $=n(n-2)$.

I. n even. For A the sum of the areas is the area of the polygon $=\frac{1}{2}nr^2\sin(2\pi/n)$; for B and J combined $nr^2\sin(2\pi/n)=$sum of areas; for C and I, the sum is $2nr^2\sin(2\pi/n)-4(n-2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$; for D and H, $3nr^2\sin(2\pi/n)-4nr^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)-4(n-4)r^2\sin(\pi/n)\times\sin(2\pi/n)\sin(3\pi/n)$. For the next pair of vertices the sum is



n even.



n odd.

$$4n^2\sin(2\pi/n)-4(n+2)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-4(n-2)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-4(n-6)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n).$$

The sum of the areas of the triangles for $\frac{n-2}{2}$th, the $\frac{n}{2}$th and the $\frac{n+2}{2}$th vertices are equal, and their combined sum for the three vertices is

$$\frac{3(n-2)}{4}r^2\sin(2\pi/n)-6(2n-8)r^2\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-6(2n-12)r^2\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)$$
$$-6(2n-16)r^2\sin(\pi/n)\sin(3\pi/n)\sin(4\pi/n)$$
$$-\ldots-6(4)\,r^2\sin(\pi/n)\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi.$$

The total sum for all is

$$[\tfrac{1}{2}n+n(1+2+3+\ldots+\frac{n-2}{2})\ +\frac{n(n-2)}{4}]r^2\sin(2\pi/n)$$
$$-[4r^2(n-2+n+n+2+n+4+\ldots+2n-8)$$
$$+2r^2(2n-8)]\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-[4r^2(n-4+n-2+n+n+2+\ldots+2n-12)$$
$$+2r^2(2n-12)]\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)-\text{etc.},$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-3r^2\sin(\pi/n)[(n-2)(n-4)\sin(\pi/n)\sin(2\pi/n)$$
$$+(n-4)(n-6)\sin(2\pi/n)\sin(3\pi/n)+(n-6)(n-8)\sin(3\pi/n)\sin(4\pi/n)+\ldots$$
$$+8\sin\frac{n-4}{2n}\pi\sin\frac{n-2}{2n}\pi]$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-24r^2[\cos(\pi/n)\cos(2\pi/n)+3\cos(2\pi/n)\cos(3\pi/n)$$
$$+6\cos(3\pi/n)\cos(4\pi/n)+10\cos(4\pi/n)\cos(5\pi/n)+\ldots$$
$$+\frac{(n-2)(n-4)}{8}\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]\sin(\pi/n)$$
$$=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-12r^2\sin(\pi/n)\cos(\pi/n)(1+3+6+10+\ldots+\frac{(n-2)(n-4)}{8}]$$
$$-12r^2\sin(\pi/n)[\cos(3\pi/n)+3\cos(5\pi/n)+6\cos(7\pi/n)+\ldots$$
$$+\frac{(n-2)(n-4)}{8}\cos\frac{n-3}{n}\pi]=\tfrac{1}{8}n^3r^2\sin(2\pi/n)-\tfrac{1}{8}n(n-2)(n-4)r^2\sin(2\pi/n)$$
$$+\frac{3n[(n-2)\cos(2\pi/n)-(n-4)]r^2\sin(2\pi/n)}{8[\sin(\pi/n)]^2}$$
$$=\frac{r^2n\{(3n-2)[\sin(\pi/n)]^2-1\}\cos(\pi/n)}{2\sin(\pi/n)}$$

$\therefore$ Average area is $\dfrac{r^2\cos(\pi/n)}{2(n-2)\sin(\pi/n)}\{[3n-2][\sin(\pi/n)]^2-1\}$.

II. n odd. By a similar process we easily get for the total sum

$$[\tfrac{1}{2}nr^2+nr^2(1+2+3+4+\ldots+\frac{n-1}{2})]\sin(2\pi/n)$$
$$-4r^2(n-2+n+n+2+n+4+\ldots+2n-7)\sin(\pi/n)\sin(\pi/n)\sin(2\pi/n)$$
$$-4r^2(n-4+n-2+n+n+2+\ldots+2n-11)\sin(\pi/n)\sin(2\pi/n)\sin(3\pi/n)\ -\text{ etc.},$$
$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-3r^2\sin(\pi/n)[(n-1)(n-3)\sin(\pi/n)\sin(2\pi/n)$$
$$+(n-3)(n-5)\sin(2\pi/n)\sin(3n/\pi)+(n-5)(n-7)\sin(3\pi/n)\sin(5\pi/n)+\ldots$$
$$+8\sin\frac{n-3}{2n}\pi\sin\frac{n-1}{2n}\pi]$$
$$=\frac{n^3+3n}{8}r^2\sin(2\pi/n)-24r^2\sin(\pi/n)[\cos(\pi/2n)\cos(3\pi/n)$$
$$+3\cos(3\pi/n)\cos(5\pi/n)+6\cos(5\pi/n)\cos(7\pi/n)+\ldots$$
$$+\frac{(n-1)(n-3)}{8}\cos\frac{n-4}{2n}\pi\cos\frac{n-2}{2n}\pi]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$
$$-12r^2\sin(\pi/n)\cos(\pi/n)[1+3+6+10+\ldots+\frac{(n-1)(n-3)}{8}]$$
$$-12r^2\sin(\pi/n)[\cos(2\pi/n)+3\cos(4\pi/n)+6\cos(6\pi/n+\ldots$$
$$+\frac{(n-1)(n-3)}{8}\cos\frac{(n-3)\pi}{n}]=\frac{n^3+3n}{8}r^2\sin(2\pi/n)$$
$$-\tfrac{1}{8}(n+1)(n-1)(n-3)r^2\sin(2\pi/n)+\frac{3r^2}{2\sin(\pi/n)}$$
$$-\frac{[3(n^2-2n-3)\sin(2\pi/n)-3(n^2-1)\sin(2\pi/n)\cos(2\pi/n)]r^2}{8[\sin(\pi/n)]^2}$$
$$=\frac{r^2}{4\sin(\pi/n)}\{(4n+3-3n^2)[\sin(\pi/n)]^2\cos(\pi/n)+6+6(n+1)\cos(\pi/n)\}.$$

The average area is

$$\frac{r^2}{4n(n-2)\sin(\pi/n)}\{6+6(n+1)\cos(\pi/n)-(3n^2-4n-3)\cos(\pi/n)[\sin(\pi/n)]^2\}.$$

For the foregoing solutions the following explanation is necessary.

$$2\sin\frac{(p-q)\pi}{n}\sin\frac{(p-q+1)\pi}{n}=\cos(\pi/n)+\cos\frac{(2p-2q+1)\pi}{n}$$

$$2\cos\frac{(p-q)\pi}{2n}\cos\frac{(p-q+2)\pi}{2\pi}=\cos(\pi/n)+\cos\frac{(p-q+1)\pi}{n}$$

Let $C=\cos3\theta+3\cos5\theta+6\cos7\theta+\ldots+\frac{(n-2)(n-4)}{8}\cos(n-3)\theta,$

$$S = i\sin 3\theta + 3i\sin 5\theta + 6i\sin 7\theta + \ldots + \frac{(n-2)(n-4)}{8} i\sin(n-3)\theta,$$

where $\theta=\pi/n$, $i=\sqrt{-1}$.

$$\therefore C+S = \cos 3\theta + i\sin 3\theta + 3(\cos 5\theta + i\sin 5\theta) + 6(\cos 7\theta + i\sin 7\theta) + \ldots$$

$$+\frac{(n-2)(n-4)}{8}[\cos(n-3)\theta + i\sin(n-3)\theta] = (\cos\theta + i\sin\theta)^3 + 3(\cos\theta + i\sin\theta)^5$$

$$+6(\cos\theta + i\sin\theta)^7 + \ldots + \frac{(n-2)(n-4)}{8}(\cos\theta + i\sin\theta)^{n-3} = e^{3i\theta} + 3e^{5i\theta} + 6e^{7i\theta} + \ldots$$

$$+\frac{(n-2)(n-4)}{8}e^{(n-3)i\theta} = y^3[1+3y^2+6y^4+10y^6+\ldots+\frac{(n-2)(n-4)}{8}y^{n-6}]$$

$$=\frac{y^3}{(1-y^2)^3} - \frac{(n-\;)(n-4)y^{n+3} + n(n-2)y^{n-1} - 2n(n-4)y^{n+1}}{8(1-y^2)^3}$$

Putting $y=e^{i\theta}=\cos\theta+i\sin\theta$, and equating C=the rational part, we get

$$C = \frac{2n(n-4)\sin(2\pi/n) - n(n-2)\sin(4\pi/n)}{64[\sin(\pi/n)]^3}.$$

MISCELLANEOUS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the number of real roots of the equation $100\sin x = x$, and show the largest root is approximately 96.10. Find $\tan 39°$ to three places of decimals. How many real roots of $\tan x = 1/x^2$ lie between 0 and 2π?

Solution by HENRY HEATON, Belfield, N. D.

(1). The equation may be written $x/\sin x = 100$, As x varies from 0 to $\frac{1}{2}\pi$, $x/\sin x$ takes the successive values between 1 and $\frac{1}{2}\pi$. Hence the equation has no real root in the first quadrant. As x varies from $\frac{1}{2}\pi$ to π, $x/\sin x$ takes all the successive values from $\frac{1}{2}\pi$ to ∞. Hence the equation has one real root in the second quadrant. In the third and fourth quadrants $x/\sin x$ is negative. Hence there can be no real root. At every round as x passes through the first and second quadrant $x/\sin x$ varies from ∞ to a minimum value then back to ∞. This minimum value is in the first quadrant when $x=\tan x$. After the first round as long as $x<100$ there are two real positive roots to every round. Hence there can be no real positive root when $x>31\pi$, and the whole number of real positive roots is 31. The number of real negotive roots is evidently the same with the same numerical values. Hence the whole number of real roots is 62. The largest real root evidently occurs between $x=30\frac{1}{2}\pi$ and 31π. Put $x=y+30\frac{1}{2}\pi$. Then $\sin(y+30\frac{1}{2}\pi)=\frac{1}{100}(y+30\frac{1}{2}\pi)$.

$\therefore \cos y = .958185 + \frac{1}{100}y$.

If $\cos y = .95818$, $y=16°\ 38' = .2903^{(r)}$, a first approximate value of y. If $\cos y = .95818 + .002903 = .96108$, $y = 16°\ 2' = .27983^{(r)}$, a second approximation.